IMPULSE ACTUATED TIME-OPTIMAL OPERATIONS IN SECOND-ORDER LINEAR SYSTEMS

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In the present work the authors continue the investigations [1 to 6] of the time-optimal operations in linear systems with constant coefficients. The control vector is assumed to be one-dimensional and bounded with respect to the "impulse" [6]. This means that the time integral of the modulus of the control vector does not exceed some positive constant M. Second order systems are investigated and the conditions of existence of time-optimal operations (*) between the points (x_{10}, x_{20}) and (x_1, x_2) of the phase plane, are determined. We show that, when these conditions are satisfied, then an operation of duration $T(x_{10}, x_{20}, x_1, x_2)$ is actuated by the impulses $\mu_1(x_{10}, x_{20}, x_1, x_2)$ and $\mu_2(x_{10}, x_{20}, x_1, x_2)$, the number of which is not greater than two. Conditions of continuity and differentiability of the functions T, μ_1 and μ_2 and of the functions $t^1(x_{10}, x_{20}, x_1, x_2)$ and $t^2(x_{10}, x_{20}, x_1, x_2)$ defining the relation between the time of appearance of the impulses μ_1 and μ_2 and the coordinates of the initial and final points are derived; a geometrical interpretation is also given.

1. Let the system of equations

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1u_1, \qquad \frac{dx_2}{dx} = a_{21}x_1 + a_{22}x_2 + b_2u_1$$

be given, where u_1 is a scalar control. Directing the x_2-axis of the basic coordinate system along the vector $b_1\mathbf{i} + b_2\mathbf{j}$, we obtain, using the same notation,

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2, \qquad \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2u_1$$

If $a_{12} \neq 0$, then differentiating the first equation with respect to time and eliminating x_2 and dx_2/dt, we obtain

$$\frac{d^2x_1}{dt^2} + b\frac{dx_1}{dt} + cx_1 = du_1$$

If, on the other hand, $a_{12} = 0$, then the above elimination is impossible. In this case however, x_1 is independent of u_1, and the system is under

*) From now on the term time - optimal operation will also be referred to as "operation".

incomplete control [3]. Let us assume that the system is under complete control, and introducing the notation

$$x = x_1, \quad x^{\cdot} = dx_1 / dt, \quad du_1 = u, \quad x^{\cdot\cdot} = dx^{\cdot} / dt$$

we arrive at

$$x^{\cdot\cdot} + bx^{\cdot} + cx = u \tag{1.1}$$

for which we shall seek, at first, the operations from the point (0, 0) to the point $(x_0, x_0^{\cdot})$ on the class of all possible scalar controls with integrable modulus, subject to the condition

$$\int_0^\infty |u|\,dt \leqslant 1 \tag{1.2}$$

Let us consider the homogeneous equation

$$x^{\cdot\cdot} + bx^{\cdot} + cx = 0$$

and its normal system of independent solutions

$$x = c_1\varphi_1(t) + c_2\varphi_2(t), \qquad \varphi_1(0) = 1, \qquad \varphi_2(0) = 0$$
$$x^{\cdot} = c_1\varphi_1^{\cdot}(t) + c_2\varphi_2^{\cdot}(t), \qquad \varphi_1^{\cdot}(0) = 0, \qquad \varphi_2^{\cdot}(0) = 1$$

In [6] the author discussed time-optimal operations from the point $(x_0, x_0^{\cdot})$ to the point (0, 0). Rephrasing the conditions of existence of these operations [6] for the problem of striking the point $(x_0, x_0^{\cdot})$ from the point (0, 0), we obtain a theorem.

T h e o r e m 1.1. Time-optimal operation between the points (0, 0) and $(x_0, x_0^{\cdot})$ exists if and only if numbers c_1°, c_2° and $T > 0$ exist, which solve the problem

$$\min_{c_1, c_2} [\max |c_1\varphi_2(T - t) + c_2\varphi_2^{\cdot}(T - t)|] = 1 \quad (0 \leqslant t \leqslant T)$$
$$c_1x_0 + c_2x_0^{\cdot} = -1 \tag{1.3}$$

If c_1°, c_2° and $T > 0$ solve the problem (1.3) and T has the smallest possible value, then T is optimal, the optimal control u° is impulsive, and is given by Formula

$$u^\circ = \mu_1\delta(t - t^1) + \ldots + \mu_n\delta(t - t^n)$$

where $\delta(t - t^i)$ is the impulse δ-function, and $t^1, \ldots, t^n$ are roots of

$$|c_1^\circ\varphi_2(T - t) + c_2^\circ\varphi_2(t - T)| = 1$$

Sum of moduli of control impulses μ_i reaches its maximum

$$|\mu_1| + \ldots + |\mu_n| = 1$$

for all finite points, except the points lying on the open interval $x_0 = 0$, $-1 < x_0^{\cdot} < 1$. The latter are possible terminal points for a fast operation with the time $T = 0$, and $|\mu_1| < 1$, $|\mu_2| = 0$.

Obviously, an arbitrary time-optimal operation commencing at zero, must be actuated by a nonzero impulse μ_1. Consequently, $t^1 = 0$, and

$$|c_1\varphi_2(T) + c_2\varphi_2^{\cdot}(T)| = 1$$

is fulfilled.

Let us now disregard, for the time being, the second equation of (1.3) and putting $T > 0$, let us look for the solutions c_1°, c_2°, t^2, ..., t^n of the problem (1.4)

$$\max |c_1\varphi_2(T-t) + c_2\varphi_2^{\cdot}(T-t)| = 1, c_1\varphi_2(T) + c_2\varphi_2^{\cdot}(T) = \pm 1 \quad (0 \leqslant t \leqslant T)$$

If they solve the above problem when the right-hand side of the second equation of (1.4) is equal to -1, then $-c_1^\circ$, $-c_2^\circ$, t^2, ..., t^n solve the problem for $+1$. Let us therefore consider the case of -1 only. With this assumption, a constant $c_3(c_1, c_2, T)$ always exists such, that the identity in t

$$c_1\varphi_2(T-t) + c_2\varphi_2^{\cdot}(T-t) \equiv -\varphi_2^{\cdot}(-t) + c_3\varphi_2(-t)$$

is fulfilled.

Let c_3, t^2, ..., t^n be the solution of the problem

$$\max |-\varphi_2^{\cdot}(-t) + c_3\varphi_2(-t)| = 1 \quad (0 \leqslant t \leqslant T) \tag{1.5}$$

Then Equations

$$c_1\varphi_2(T) + c_2\varphi_2^{\cdot}(T) = 1, \qquad |c_1\varphi_2(T-t^i) + c_2\varphi_2^{\cdot}(T-t^i)| = 1$$

can be used to find c_1° and c_2°.

2. We shall attempt to solve the problem (1.5) by assigning various complex values to the roots λ_1 and λ_2 of the characteristic polynomial of (1.2).

1) The roots are complex, with positive real parts

$$\lambda_{12} = \beta \pm i\omega, \qquad 2\beta = -b > 0, \qquad 0 < b^2 < 4c$$

In this case it is convenient to replace c_3 with the phase shift φ

$$c_1\varphi_2(T-t) + c_2\varphi_2^{\cdot}(T-t) = a(c_3, t) = -\exp(-\beta t)\cos(\varphi - \omega t)(\cos\varphi)^{-1} \tag{2.1}$$

Since the function $a(c_3, t)$ cannot be less than -1 when t increases, then $a^{\cdot}(c_3, 0) \geq 0$. This limits φ in the following manner:

$$-\tfrac{1}{2}\pi \leq \varphi \leq \tan^{-1}(\beta/\omega)$$

For any admissible φ, the nearest maximum of the function $a(\varphi, t)$ is reached when $t(\varphi)$ satisfies

$$\omega t(\varphi) = \pi + \varphi - \tan^{-1}(\beta/\omega)$$

Value of the maximum of $\mu(\varphi)$ of the function $a(\varphi, t(\varphi))$ is given by

$$\mu(\varphi) = \exp[(\beta/\omega)(\pi + \varphi - \tan^{-1}(\beta/\omega))](\cos\varphi)^{-1}\cos[\tan^{-1}(B/\omega)] \tag{2.2}$$

a direct check that $\mu(\varphi)$ increases as φ decreases and will be always positive over the variation of φ within $-\frac{1}{2}\pi < \varphi < \tan^{-1}(\beta/\omega)$, is easy. From (2.1) we see, that if $\mu(\varphi) < 1$, then $|a(\varphi, t)| < 1$ for all $t > 0$. Hence the solution of our problem must be a function, the maximum of which $\mu(\varphi) \geq 1$.

Let us now consider $a(\varphi_0, t)$ such, that $\mu(\varphi_0) = 1$. Then, putting $t(\varphi_0) = t_2$ we obtain the identity

$$a(\varphi_0, t) \equiv \varphi_1(t_2 - t)$$

Assuming $t = 0$, we have

$$\varphi_1(t_2) = -1 \tag{2.3}$$

From the above we see, that t_2 is the smallest positive root of (2.3), which obviously always exists since $\beta > 0$, and when $t(\varphi) < t_2$, $\max\mu(\varphi) > 1$. This means, that for any $T < t_2$, variation in φ will produce $a(\varphi, T) = 1$. If, on the other hand $T > t_2$, then $a(\varphi_0, t)$ is the only solution to the problem (1.2). Returning to the initial notation $a(c_1, c_2, t)$, we obtain the result.

If $T \le t_2$, then t^2 defines the instant of the second impulse which is equal to T, and

$$c_1^\circ\varphi_2(0) + c_2^\circ\varphi_2'(0) = c_2^\circ = 1$$

If $T > t_2$, then $t^2 = t_2$, while $c_1\varphi_2(T - t_2) + c_2\varphi_2'(T - t_2) = 1$.

If the roots are real and positive, then the situation is exactly the same, and the proof is analogous.

2) Let the complex roots have negative real parts

$$\lambda_{1,2} = \beta \pm \omega i, \ \beta < 0$$

From (2.1) and (2.2) it follows, that for any admissible value of $\varphi \le \tan^{-1}(\beta/\omega)$, $\max \mu(\varphi) > 1$. This means that all the admissible functions are monotonous and t^2 can only be equal to T. In the present case however, the instant t^2 is bounded from above and corresponds to the smallest admissible maximum $\mu_1(\tan^{-1}(\beta/\omega))$, for this value of the angle, $a'(\varphi_0, 0) = 0$, hence $a(\varphi_0, t) \equiv -\varphi_1(-t)$.

Consequently, the maximum admissible value of $T = t_2$ can be found from

$$\varphi_1(-t_2) = -1 \tag{2.4}$$

and t_2 will be the smallest positive root of this equation.

This means that for $T \le t_2$, the instant of the second impulse $t^2 = T$, and $c_2 = 1$. If $T > t_2$, then the problem (1.5) has no solution. The same result is obtained if $\lambda_2 < \lambda < 0$ are real numbers.

3) Let the roots differ in sign, and be different from zero, $\lambda_1 > 0$, $\lambda_2 < 0$.

In this case

$$a(t, c_3) = \frac{-\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} + c_3 \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}$$

Condition $a'(0, c_3) \ge 0$ results in $\lambda_1 + \lambda_2 - c_3 \ge 0$.

The derivative $a'(t, c_3)$ can become equal to zero for $t > 0$ only, when

$$e^{(\lambda_1 - \lambda_2)t} = \frac{\lambda_1^2 - c_3\lambda_1}{\lambda_2^2 - c_3\lambda_2} > 1$$

This inequality is fulfilled if and only if $c_3 < \lambda_2$, in which case $a(t, c_3)$ has a maximum $\mu(c_3)$ at the point $t = t(c_3)$, and

$$a(t, c_3) = \mu(c_3)\varphi_1(t(c_3) - t)$$

is true.

The condition of maximum of $a''(t(c_3), c_3) \le 0$, gives $-\mu(c_3)c \le 0$. Since $c = \lambda_1\lambda_2 < 0$ then $\mu(c_3) \le 0$. This maximum is unique and when $t > t(c_3)$, the function $a(t, c_3)$ tends to $-\infty$.

If $c_3 > \lambda_2$, then $a(t, c_3)$ increases monotonously to ∞. Obviously, at any T, time of the second impulse t^2 can only be equal to T, and for any T, two values $c_3^1 > \lambda_2$ and $c_3^2 < \lambda_2$ can be selected, such that $a(T, c_3^1) = +1$, corresponds to the first value, and $a(T, c_3^2) = -1$ to the second.

Returning to the initial constants c_1 and c_2, we obtain $t^2 = T$, $c_2^1 = +1$, $c_2^2 = -1$.

4) Let the roots $\lambda_{12} = \pm i\omega$ be purely imaginary.

Reasoning as in (1), we obtain that, when $T < \pi/\omega = t_2$, the solution of (1.4) is of the form $c_2 = +1$, $t^2 = T$. If $T \ge \pi/\omega$, then some other solutions may appear. Below we shall show that operations with time $T \ge \pi/\omega$ do not exist.

5) Let $\lambda_1 > 0$, $\lambda_2 = 0$, then, we have

$$a(t,\ c_3) = e^{-\lambda_1 t}\left(\frac{c_3}{\lambda_1} - 1\right) - \frac{c_3}{\lambda_1}$$

Condition $a^{\cdot}(0, c_3) \geq 0$ gives $c_3 \leq \lambda_1$ and $a(t, c_3)$ becomes a nondecreasing function with the limit value of c_3/λ_1 as $t \to \infty$. Consequently, $c_3 < -\lambda_1$ can always be expressed so, as to have $a(T, c_3) = 1$. This will be the first solution of the problem. Second solution $c_3 = \lambda_1$ gives $a(t, c_3) \equiv -1$.

Hence, the first solution is $t^2 = T$, $c_2 = 1$, while the second one is $c_2 = -1$ $(0 < t^2 \leq T)$, and the instant t^2 remains undefined.

6) Let $\lambda_2 < 0$, $\lambda_1 = 0$. Using the same scheme, we reach the conclusion that two solutions of (1.4) are possible. They are

$$c_2 = 1,\quad t^2 = T \text{ (First solution) } c_2 = -1,\quad 0 < t^2 \leqslant T \text{ (Second solution)}$$

7) Let $\lambda_1 = \lambda_2 = 0$. Then

$$c_2 = 1,\quad 0 < t^2 \leqslant T \text{ (First solution) } c_2 = -1,\quad 0 < t^2 \leqslant T \text{ (Second solution)}$$

In the following we shall show that in cases (5), (6) and (7), time-optimal operations must have $t^2 = T$, i.e. the second impulse occurs at the last possible instant.

3. Let us now return to the problem (1.3)

$$\min_{c_1, c_2} \max |c_1\varphi_2(T-t) + c_2\varphi_2^{\cdot}(T-t)| = \min_{c_1, c_2} f(c_1, c_2, T) = +1 \quad (0 \leqslant t \leqslant T)$$
$$c_1x_0 + c_2x_0^{\cdot} = -1$$

Let $c_1^{\circ}, c_2^{\circ}, T, t^1, \ldots, t^n$ be the solution of (1.3), then for all x_0, $x_0^{\cdot}$ except $x_0 = \pm\varphi_2(T)$ and $x_0^{\cdot} = \pm\varphi_2^{\cdot}(T)$, a root t^2 exists. Indeed, if the value of unity could only be reached at $t = 0$, then such translation along the straight $c_1x_0 + c_2x_0^{\cdot} = -1$ on the c_1c_2-plane not coinciding with the straight line $c_1\varphi_2(T) + c_2\varphi_2^{\cdot}(T) = \pm 1$, could lead to $f(c_1, c_2, T) < 1$; this contradicts the the initial assumption that $c_1^{\circ}, c_2^{\circ}, T$ is the solution of (1.3).

1) Let $\lambda_{12} = \beta \pm \omega i$; $\beta > 0$, $T \leqslant t_2$, then the straight line $c_1x_0 + c_2x_0^{\cdot} = -1$ must pass through the point of intersection of

$$c_1\varphi_2(T) + c_2\varphi_2(T) = -1, \qquad c_2 = 1 \tag{3.1}$$

or through the point B of the intersection of

$$c_1\varphi_2(T) + c_2\varphi_2^{\cdot}(T) = 1, \qquad c_2 = -1 \tag{3.2}$$

Lines (3.1) and (3.2) form a parallelogram $AA'BB'$. Pencils of lines $c_1x_0 + c_2x_0^{\cdot} = 1$ passing through points A and B, have the equations

$$k(c_1\varphi_2(T) + c_2\varphi_2^{\cdot}(T) \pm 1) + (c_2 \mp 1) = 0$$

where k is the parameter of the pencil, and upper and lower signs correspond to the points A and B, respectively.

The line $c_1x_0 + c_2x_0^{\cdot} = -1$ cannot however pass through the inside of the angle $B'AA'$, since a small displacement along such a line into the parallelogram $AA'BB'$ would make $f(c_1, c_2, T) < 1$. These lines correspond to nonpositive values of k in the pencil. On the other hand, when all the points on the lines belonging to the pencil except the points A and B have nonnegative values of k, then $f(c_1, c_2, T) > 1$. From this it follows, that the solution to (1.3) for any of these lines lies either at A or at B.

The condition that $k \geq 0$ on the $x_0x_0^{\cdot}$-plane gives rise to two semi-open straight line segments

$$\frac{x_0^{\cdot}+1}{\varphi_2^{\cdot}(T)+1}=\frac{x_0}{\varphi_2(T)},\qquad \begin{matrix}0<x_0\leqslant\varphi_2(T)\\ -1<x_0^{\cdot}\leqslant\varphi_2^{\cdot}(T)\end{matrix}$$

$$\frac{x_0^{\cdot}+1}{-\varphi_2^{\cdot}(T)-1}=-\frac{x_0}{\varphi_2(T)},\qquad \begin{matrix}0<x_0\leqslant-\varphi_2(T)\\ 1>x_0^{\cdot}\geqslant-\varphi_2^{\cdot}(T)\end{matrix}$$

These segments $[a_2,\ b_1)$ and $[b_2,\ a_1)$ are shown on Fig.1, and represent the set of terminal points of operations with time T.

With T changing from zero to t_2, these segments rotate about the points b_1 and a_1, and their end-points a_2 and b_2 trace the curves $x_0=\pm\varphi_2(T)$ and $x_0^{\cdot}=\pm\varphi_2^{\cdot}(T)$, which we shall, from now on, denote by G_+ and G_-.

If $T>t_2$, then, as shown before, c_1 and c_2 can be found from

$$c_1\varphi_2(T)+c_2\varphi_2^{\cdot}(T)=-1,\qquad c_1\varphi_2(T-t_2)+c_2\varphi_2^{\cdot}(T-t_2)=+1$$

or from

$$c_1\varphi_2(T)+c_2\varphi_2\ (T)=+1,\qquad c_1\varphi_2(T-t_2)+c_2\varphi_2\ (T-t_2)=-1$$

Using analogous considerations, we obtain two other segments $[a_4b_5)$ and $[b_4a_5)$ representing geometrical loci of the end-points of an operation of duration $T>t_2$. These segments are tangent to G_- and G_+ at the points b_5 and a_5.

It is easy to see from the geometry of the system that, whatever is the point $x_0,\ x_0^{\cdot}$, it can, for some T, be found either on the segments $[a_2b_1)$ and $[b_2a_1)$, or on $[a_4b_5)$ and $[b_4a_5)$. Hence, time-optimal operation always exists. Time $T\leq t_2$ and the impulses μ_1 and $\mu_2=\pm(1-|\mu_1|)$ can be found from

$$(x_0\ \mathrm{sign}\ x_0)=\mu_1\varphi_2(T),\qquad (x_0^{\cdot}/\ \mathrm{sign}\ x_0)=\mu_1(\varphi_2^{\cdot}(T)+1)-1 \tag{3.3}$$

If on the other hand $T>t_2$, then we have

$$\begin{aligned}\frac{x_0\varphi_2(t_2-T)+x_0^{\cdot}\varphi_2^{\cdot}(t_2-T)}{\mathrm{sign}\,[x_0\varphi_2(t_2-T)+x_0^{\cdot}\varphi_2^{\cdot}(t_2-T)]}&=\mu_1\varphi_2(t_2)\\ \frac{x_0\varphi_2^{\cdot}(t_2-T)+x_0^{\cdot}\varphi_2^{\cdot}(t_2-T)}{\mathrm{sign}\,[x_0\varphi_2(t_2-T)+x_0^{\cdot}\varphi_2^{\cdot}(t_2-T)]}&=\mu_1(\varphi_2^{\cdot}(t_2)+1)-1\end{aligned} \tag{3.4}$$

For any T, the curves G_+ and G_- together with segments $[a_2b_1)$ and $[b_2a_1)$ or with $[a_4b_5)$ and $[b_4a_5)$, define a convex region $D(T)$, which shall be called the region of attainability. We shall show later that during the time $t\leq T$, no admissible control can steer the operation across the boundary of this region, but any interior or boundary point of $D(T)$ can be reached.

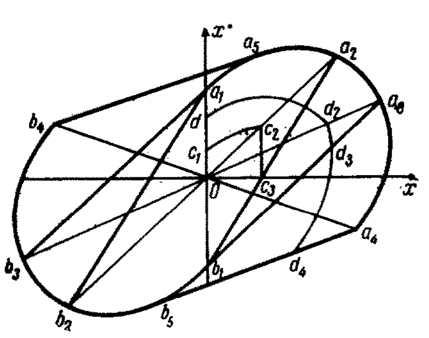

Fig. 1

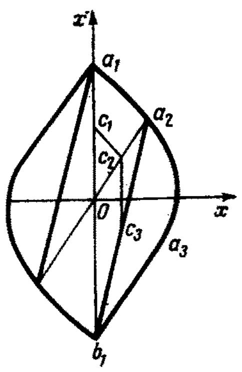

Fig. 2

On Fig.1, the curve $oc_1c_2c_3$ represents an operation (this notation is also used to describe a curve representing the operation on Figs.2 and 7) into the points c_3 in time $T<t_2$, while $od_1d_2d_3$ represents an operation into d_3 in time $T>t_2$.

2) Let $\lambda_{1,2} = \beta \pm i\omega$, $\beta < 0$. Considerations similar to the previous ones give segments $[a_2 b_1)$ and $[b_2 a_1)$ with fixed end-points b_1 and a_1, the other end-points a_2 and b_2 sliding along the curves G_+ and G_- together with the image point (Fig.2). In this case however, changes of the region $D(T)$ will terminate at the instant t_2. We should note that t_2 is the smallest positive root of Equation $\varphi_1(-t_2) = -1$. Region $D(t_2)$ represents a set of terminal points of all possible operations. No operation terminating in a point outside the region $D(t_2)$ is possible and it will be shown below, that such point is completely unattainable. Time T and the impulse μ_1 of the operation terminating at the point belonging to $D(t_2)$ are found from Equations (3.3).

3) Let $\lambda_1 > 0$, $\lambda_2 < 0$. In this case all four vertices of the parallelogram on the $c_1 c_2$ plane formed by the lines

$$c_1\varphi_2(T) + c_2\varphi_2(T) = \pm 1, \qquad c_2 = \pm 1$$

are admissible to the line $c_1 x_0 + c_2 x_0^{\cdot} = -1$.

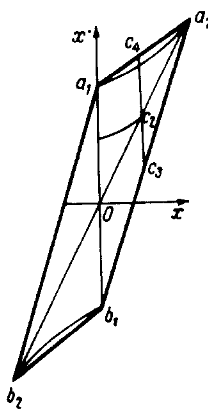

Fig. 3

Admissible orientations of this line at the vertices A and B, give rise to the segments $[a_2 b_1)$ and $[b_2 a_1)$ of the $xx^{\cdot}$ plane, while at A' and B' they give rise to $(a_1 a_2]$ and $(b_1 b_2]$. These segments form, on the $xx^{\cdot}$ plane, a parallelogram $a_1 a_2 b_1 b_2$ (Fig.3).

Time T and the impulse μ_1 are given by (3.3), however the sign of the impulse μ_2 acting on the point x_0, $x_{01}^{\cdot}$ of $[a_2 b_1)$ differs from that of μ_2' acting on the point x_0, $x_{02}^{\cdot}$ of $(a_1 a_2]$. When $T \to \infty$, the parallelogram becomes the interior of a strip bounded by the lines $x^{\cdot} \pm 1 = \lambda_1 x$.

4) If $\lambda_{12} = \pm i\omega$, then for $T < \pi/\omega$ we obtain, in the analogous manner, region $D(T)$ bounded by elliptic arcs

$$G_{+,-}\ [x = \pm \sin\omega\tau,\ \ x^{\cdot} = \pm\omega\cos\omega\tau;\ \ 0 \leqslant \tau \leqslant T]$$

(Fig.4) and segments $[a_2 b_1)$ and $[b_2 a_1)$. When $T = \pi/\omega$, the ellipse becomes complete. Obviously, T is always smaller than π/ω, since any point inside the ellipse can be reached in the time $t < \pi/\omega$. Functions T and μ_1 are given by (3.3).

5) Let $\lambda_1 > 0$, $\lambda_2 = 0$. As before, we obtain the segments $[a_2 b_1)$ and $[b_2 a_1)$. The second solution of (1.5), $c_2 = -1$ in which c_1 and t^2 are undefined, remains to be investigated. The line $c_1 x_0 + c_2 x_0 = -1$ cannot intersect the line $c_2 = -1$, because at $c_2 = -a > -1$, we have $f(c_2, T) < 1$. This means that $x_0 = 0$ and $x_0^{\cdot} = 1$, and an operation terminating at this point has zero duration. Hence, second solution does not give any new operations (Fig.5).

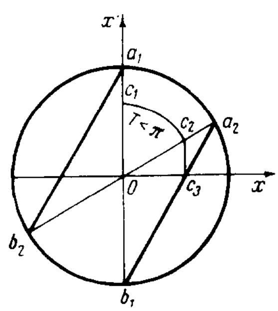

Fig. 4

The parallelogram $a_1 a_2 b_1 b_2$ is the region of attainability $D(T)$ and as $T \to \infty$, it becomes a strip $x^{\cdot} - 1 \leqslant \lambda_1 x \leqslant x^{\cdot} + 1$.

6) If $\lambda_2 < 0$, $\lambda_1 = 0$, then the parallelogram $a_1 a_2 b_1 b_2$, is also the region $D(T)$. Second solution of (1.5) $c_2 = -1$ does not lead to any new operations. As $T \to \infty$, it becomes a parallelogram $a_1 a_3 b_1 b_3$ (Fig.6) with vertices $a_3 b_3$ on the x-axis.

7) If $\lambda_1 = \lambda_2 = 0$, then the sides $a_1 a_2$ and $b_1 b_2$ of $a_1 a_2 b_1 b_2$ are parallel to the x-axis. From the geometry it is clear, that the operations terminating at the points of the segments $[a_2 b_1)$ and $[b_2 a_1)$ must have $t^2 = T$. As $T \to \infty$, the parallelogram becomes a strip $-1 \leq x^{\cdot} \leq +1$ (Fig.7). In cases (5), (6) and (7), T and μ_1 are, of course, given by (3.3).

4. Before beginning to discuss the continuity and differentiability of the time of duration of the operation $T(x_0^{\cdot}, x_0)$, of the first impulse $\mu_1(x_0, x_0^{\cdot})$ and of the instants $t^1(x_0, x_0^{\cdot})$, $t^2(x_0, x_0^{\cdot})$, assumed to be the functions of the terminal point, we should note that $t^1 = 0$ and $t^2 = T$, since $t^2 = t_2$. We shall therefore concentrate our attention on the functions $T(x_0, x_0^\cdot)$ and $\mu_1(x_0, x_0^\cdot)$. We shall divide all the points x_0, $x_0^\cdot$ into the groups of interior, boundary and exterior points. The open segment $x_0 = 0$, $-1 < x_0^\cdot < +1$ will correspond to the boundary points together with $G_\pm$ in case (1), with the boundary of $D(t_2)$ in case (2), the lines $x_0^\cdot \pm 1 = \lambda_1 x$ in case (3), the boundary of $D(\pi/\omega)$ in case (4), the lines $x_0^\cdot \pm 1 = \lambda_1 x$ in case (5), the boundary of $D(\infty)$ in case (6) and the lines $x_0^\cdot = \pm 1$ in case (7). All points which, for some T , lie within $D(T)$ will correspond to interior points and all the remaining ones – to the exterior points.

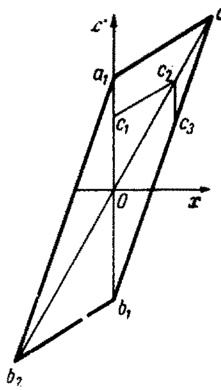

Fig. 5

First we shall consider the interior point $x_0^\cdot$, x_0. In all cases (except (1) when $T > t_2$) the operation can be determined by (3.3). Assuming for definiteness that $x_0 > 0$ in (3.3) and differentiating both parts, we obtain

$$dx_0 = \varphi_2(T)d\mu_1 + \mu_1\varphi_2^\cdot(T)dT,$$
$$dx_0^\cdot = [1 + \varphi_2^\cdot(T)]d\mu_1 + \mu_1\varphi_2^{\cdot\cdot}(T)dT \qquad (4.1)$$

Assuming that (4.1) represents a system with unknowns $d\mu_1$ and dT we note, that its determinant is equal to zero if and only if a constant e exists, satisfying the equalities

$$e\varphi_2(T) + \varphi_2^\cdot(T) = -1, \qquad e\varphi_2^\cdot(T) + \varphi_2^{\cdot\cdot}(T) = 0$$

Using the argument similar to that of Section 2 we find, that T satisfies Equation $\varphi_1(-T) = -1$. This can only happen in case (2) when $t_2 = T$. Since the point is interior, the last equation is impossible, hence, the determinant is not equal to zero. Consequently, the operation defined by (3.3) exists in some neighborhood of x_0, $x_0^\cdot$. This means that for any interior point, partial differentials of T and μ_1 exist, and are defined by (4.1).

Analogous consideration for the case (1), $T > t_2$ leads to the same conclusion. Next we shall consider the boundary points. Passing to the limit in (3.3) as $x_0 \to \pm 0$, $x_0^\cdot \to a$, $(0 < a < 1)$, gives, in the limit, $T = 0$, $\mu_1 = \frac{1}{2}(1 \pm a)$. Substituting these values into (4.1) and remembering that the signs of the left-hand sides should be changed as $x_0 \to -0$, we obtain the limiting values for partial derivatives as $x_0 \to \pm 0$

$$\frac{\partial T}{\partial x_0} = \pm\frac{2}{1 \pm a}, \quad \frac{\partial T}{\partial x_0^\cdot} = 0, \quad \frac{\partial \mu_1}{\partial x_0} = \pm\frac{b}{2}, \quad \frac{\partial \mu_1}{\partial x_0^\cdot} = \pm\frac{1}{2}$$

Case $\lambda_{12} = \beta \pm i\omega$ $(\beta > 0)$; also, we shall assume that the point $(x_0, x_0^\cdot)$ coincides with the point b_5 (Fig.1) of the boundary curve G_- . Putting now $T_+ = \lim T$ when $(x_0, x_0^\cdot) \to b_5^+$ from the direction of the origin of coordinates and $T_- = \lim T$ when $(x_0, x_0^\cdot) \to b_5^-$ from the outside, we see directly from (Fig.1) that $T_- - T_+ = t_2$ .The derivative of T in the direction $b_5 a_1$ is equal to zero, and the derivative in the direction of the tangent to G_- , equals unity.

Angle α between the tangent and the line $b_5 a_1$, is always less than π, hence the inner normal derivative satisfies Equation

$$\left(\frac{\partial T}{\partial n}\right)_+ \cos\alpha - \sin\alpha = 0$$

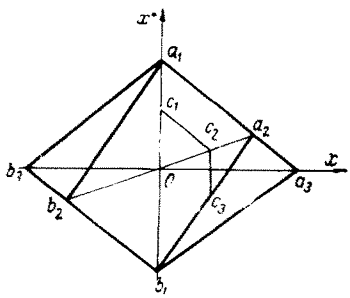

Fig. 6

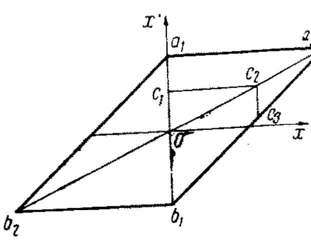

Fig. 7

and is, together with $(\partial\mu_1/\partial n)_+$, bounded. The position is different however with the bound of the outer normal derivative. Denoting by β the angle between the integral line and $b_5 a_4$, we obtain

$$\left(\frac{\partial T}{\partial n}\right)_- \sin\beta - \cos\beta = 0$$

Since $\beta \to \pi$ as $(x_0, x_0^\cdot) \to b_5^-$, the normal derivative tends to ∞, and this also applies to $(\partial\mu/\partial n)_-$.

The remaining cases fail to show any fundamental differences, therefore we shall just state the results.

C a s e $\lambda_{12} = \beta + i\omega, \beta < 0$. At all points of G_+ and G_- except $x_0 = 0$, $x_0^\cdot = \pm 1$, the inner derivatives exist and are finite. They tend to ∞ only on approach to the points a_3 and b_3, where the boundary becomes rectilinear. On the rectilinear parts of the boundary of $D(t_2)$, inner normal derivatives are infinite.

C a s e $\lambda_{12} = \pm i\omega$. Inner derivatives exist at all points of the boundary ellipse except the points $x_0 = 0$, $x_0^\cdot = \pm 1$.

C a s e $\lambda_1 > 0$, $\lambda_2 < 0$. At points $x_0 = 0$ and $x_0^\cdot = \pm 1$ derivatives do not exist. Limit values of the derivatives on approach to the points $x_0^\cdot \pm 1 = \lambda_1 x_0$ can be found from Formulas (4.1).

C a s e $\lambda_2 > 0$, $\lambda_1 = 0$. Properties of boundary derivatives follow (5) of Section 3.

C a s e $\lambda_2 < 0$, $\lambda_1 = 0$. Properties of boundary derivatives follow (6) of Section 3.

C a s e $\lambda_1 = 0$, $\lambda_2 = 0$. In this case μ_1 and T are explicitly given by

$$\mu_1 = \left(\frac{1 - x_0^\cdot}{2}\right) \operatorname{sign} x_0, \qquad T = \frac{2\,|x_0|}{|1 + x_0^\cdot \operatorname{sign} x_0|}$$

which describe all their properties.

5. Two important remarks must be made. A hit from the point (0, 0) to the point $(x_0, x_0^\cdot)$ in time T_1 is possible if and only if [8]

$$\min_{c_1, c_2} \max_{0 \leqslant t \leqslant T_1} |c_1\varphi_1(T_1 - t) + c_2\varphi_2(T_1 - t)| = \lambda(T_1) \geqslant 1 \tag{5.1}$$

This means that, if the hit is possible, then the decrease of T_1 to T such that $\lambda(T) = 1$ implies, that fast operation is possible. This in turn means, that the point can be hit only when a fast operation terminating at this point is possible. If $(x_0, x_0^\cdot)$ lies outside the region $\lim D(T)$ as $T \to \infty$, the hit is impossible at all.

Let the initial "reserve" be different from unity

$$\int_0^\infty |u|\,dt \leqslant M > 0, \qquad M \neq 1$$

Changing the variables $Mu_1 = u$, $Mx_1 = x$ we find, that the statement of the problem in the u_1, x_1, variables is identical to the previous one. Hence

$$\mu_1' = M\mu_1\left(\frac{x_0}{M}, \frac{x_0^\cdot}{M}\right) \qquad T' = T\left(\frac{x_0}{M}, \frac{x_0^\cdot}{M}\right)$$

and the regions of attainability can be obtained from the regions described previously, by M-tuple similarity transformation along the x- and $x^\cdot$-axes.

Now we shall consider the general case of an operation from the initial point $A(x_1, x_1^\cdot)$. A simple geometrical interpretation can be employed. Let the point A move from the initial conditions x_1, $x_1^\cdot$ along the integral line of Equation $x^{\cdot\cdot} + bx^\cdot + cx = 0$

$$x_A = x_1\varphi_1(t) + x_1^\cdot\varphi_2(t), \quad x_A^\cdot = x_1\varphi_1^\cdot(t) + x_1^\cdot\varphi_2^\cdot(t) \tag{5.2}$$

We shall consider consecutive positions of the point A with increasing t and construct, for each of these positions, a region of attainability $D_A(t)$ with its "center" at A. The set of closed regions $D_A(t)$, $0 \leq t < t_1$

also forms a closed region (*) $C(t_1)$. An operation $T(x_0, \ldots, x_1^{\cdot})$ terminating at any interior point of this region, is possible. To shed more light on the properties of the function $T(x_0, \ldots, x_1^{\cdot})$, we shall investigate the structure of the boundary of $C(t_1)$ in more detail.

At any time t_1 , the boundary of $C(t_1)$ can be composed of segments of the following types.

1°) Boundary of $D_A(0)$ is a line

$$x = x_1, \quad x_1 - 1 \leqslant x^{\cdot} \leqslant x_1^{\cdot} + 1 \tag{5.3}$$

and the boundary of $C(t_1)$ may include parts, or the whole of this line. In the following however, when the curve is stipulated, the last condition shall not be discussed.

2°) Trajectories of vertices a_1 and b_1

$$x = x_A, \qquad x^{\cdot} = x_A^{\cdot} \pm 1 \tag{5.4}$$

3°) Families of envelopes of the curvilinear segments of the boundary. Obviously, such an envelope will be the integral line of Equation $x^{\cdot\cdot} + bx^{\cdot} + cx = 0$

$$x = c_1\varphi_1 + c_2\varphi_2, \qquad x^{\cdot} = c_1\varphi_1^{\cdot} + c_2\varphi_2^{\cdot} \tag{5.5}$$

4°) Families of envelopes of the rectilinear segments of the boundary. We shall write the equation of such a segment as

$$x - x_A = \pm\lambda\varphi_2(t), \qquad x^{\cdot} - x_A^{\cdot} = \pm\lambda(\varphi_2^{\cdot}(t) + 1) \mp 1 \tag{5.6.1}$$

or in the form

$$\begin{aligned} x - x_A &= \pm\lambda\,[\varphi_2(t) - \varphi_2\,(t - t_2)] \pm \varphi_2\,(t - t_2) \\ x^{\cdot} - x_A^{\cdot} &= \pm\lambda\,[\varphi_2^{\cdot}(t) - \varphi_2^{\cdot}(t - t_2)] \pm \varphi_2^{\cdot}(t - t_2) \end{aligned} \tag{5.6.2}$$

for the case (1) when $t > t_2$. Here $0 < \lambda \leq 1$ is a positive parameter, and the first combination of signs corresponds to the segment $(b_1 a_2]$ (or $(b_5 a_4])$, while the second one to the segment $(a_1 b_2]$ (or $(a_5 b_4])$.

Differentiating (5.6.1) with respect to t with x and $x^{\cdot}$ kept constant, we obtain

$$-x_A^{\cdot} = \pm\lambda^{\cdot}\varphi_2 \pm \lambda\varphi_2^{\cdot}, \qquad -x_A^{\cdot\cdot} = \pm\lambda^{\cdot}(\varphi_2^{\cdot} + 1) \pm \lambda\varphi_2^{\cdot\cdot} \tag{5.6.3}$$

Substituting λ obtained from these equations into (5.6.1), we obtain equation of the envelope which will coincide with the boundary only when $\lambda(t)$ which is a solution of (5.6.3), satisfies the inequality $0 \leq \lambda(t) \leq 1$. This condition is necessary for the point of the envelope to be on the boundary of $D_A(t)$.

5°) Pieces of the boundary of $D_A(t_1)$.

In general, construction of the region $C(t_1)$ is difficult in the sense, that the equation of its boundary will depend on five parameters, namely λ_1 and λ_2 which are the roots of characteristic equation, x_1/M and $x_1^{\cdot}/M$ which are the normalized coordinates of the initial point, and on t_1 . With the above parameters fixed, we must construct all the curves (5.3) to (5.6), together with segments of the boundary of $D_A(t_1)$ for $0 \leq t \leq t_1$.

Some of these curves may be found to be completely inside $C(t_1)$, while others may have some of their parts lying on the boundary.

The set of these parts constitutes the boundary of the region $C(t_1)$. We should note that a segment of the boundary of $C(t_1)$ may be found inside the region $C(t_1')$ when $t_1' > t_1$. This most certainly will happen to the parts of the boundary representing segments of the boundary of $D_A(t_1)$. When $t_1' = t_1 + \Delta t$, they will completely penetrate into the region $C(t_1 + \Delta t)$ with exception perhaps, of the end-points of the sections of curves composing this segment. It is intuitively obvious, that at any interior point of this segment time of the operation will depend continuously on the initial and

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*) When $t = 0$, the region of attainability is represented by a segment $x = x_1$, $-1 + x_1^{\cdot} \leq x^{\cdot} \leq x_1^{\cdot} + 1$ and it can be regarded as a closed segment of the boundary.

final point, and will be continuously differentiable at all its interior points except the vertices.

Returning to the parts of the boundary of the type (5.3) to (5.6) we shall note, that any interior point of such a part can be found inside the region $C_1(t_1+\Delta t)$ only for some finite period of time Δt or, it may, during the whole period, remain on the boundary. From this it follows that, if at $t = t_1$ point $(x_0, x_0^{\cdot})$ is the interior point of the part of the boundary of the type (5.3) to (5.6), then time of the operation terminating at this point, exhibits a discontinuity. If the point lies at the end of one of the pieces, then the time of operation can be continuous only, when this point is also the end-point of the part of $D_A(t_1)$.

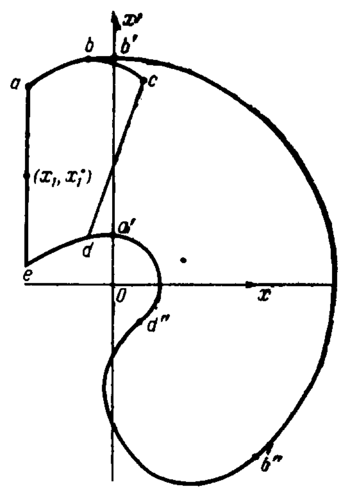

Fig. 8

As an example of construction of a region $C(t_1)$, we shall consider the case

$$\lambda_{12} = \pm i, \; x_1 < 0, \; x_1^{\cdot} > 0, \; x_1^2 + x_1^{\cdot 2} > 1.$$

The region $C_1(t_1)$ $(t_1 < \pi)$ is bounded by $aa_1a_2b_1e$ (Fig.8), while region $C_1(\pi)$ is bounded by $ab'b''d''de$.

Its boundary is composed of:

A segment ea which is $D(0)$ of the type (5.3).

Parts of circles aa_1 and eb_1 with radii $R = \sqrt{x_1^2 + x_1^{\cdot 2}}$ and centers on the axis $x = 0$, $x_{12}^{\cdot} = \pm 1$. They are the trajectories of the points a_1 and b_1, of the type (5.4).

Parts of circles $b'b''$ of radius $R_1 = R + 1$ and $d'd''$ of radius $R_2 = R - 1$ with centers at the origin. They are the envelopes of curvilinear parts of the boundary and are of the type (5.5).

Broken line $a_1a_2b_1$ which is a part of the boundary of $D_A(t_1)$ and consists of an arc of a unit circle with its center at the point $A(t_1)$ and a straight line segment a_2b_1. Last two parts are of the type (5.7).

Unit semi-circle $b''d''$ with center at $A(\pi)$. This is the part of the boundary of $D_A(t_1)$

6. Let us now construct some analytical criterions of continuity and differentiability of the functions

$$T(x_1, \ldots, x_0^{\cdot}), \; \mu_1(x_1, \ldots, x_0^{\cdot}), \; t^1(x_1, \ldots, x_0^{\cdot}), \; t^2(x_1, \ldots, x_0^{\cdot})$$

Here it will be convenient to use a moving coordinate system $AXX^{\cdot}$, its origin at the point A and its axes parallel to the axes of the basic system. The terminal point B of the operation will, with respect to these axes, move according to

$$x_B = x_0 - x_A(t), \qquad x_B^{\cdot} = x_0^{\cdot} - x_A^{\cdot}(t)$$

In the following, all coordinates and velocities of the points will be considered in the new coordinate system.

Let us introduce the velocity of a point belonging to the rectilinear part of the boundary of $D_A(t)$

$$v_1 = \pm\varphi_2^{\cdot}(t)\lambda, \qquad v_2 = \pm\varphi_2^{\cdot\cdot}(t)\lambda \qquad (0 < \lambda \leqslant 1)$$

On the curvilinear parts of the boundary, i.e. on $G_{\pm}$, we shall assume $\lambda = 1$, and we shall call the vector whose components are

$$x_C^{\cdot} = x_B^{\cdot} - v_1 - x_0^{\cdot}, \qquad x_C^{\cdot\cdot} = x_B^{\cdot\cdot} - v_2$$

the "velocity of penetration".

If the first intersection of the boundary of the region with the trajectory x_B, x_B takes place at the point $(x_E, x_E^{\cdot})$ belonging to the rectilinear segment of the boundary and the time is $t = t_1$, then the value of the parameter λ can be found from Equations

$$x_E = \pm\lambda\varphi_2(t_1), \qquad x_E^{\cdot} = \pm\lambda(\varphi_2^{\cdot}(t_1) + 1) \pm 1 \tag{6.1.1}$$

or, in case (1) of Section 3 at $t_1 > t_2$, from

$$x_E = \pm\lambda[\varphi_2(t_1) - \varphi_2(t_1 - t_2)] \pm \varphi_2(t_1 - t_2) \tag{6.1.2}$$
$$x_E^{\cdot} = \pm\lambda[\varphi_2^{\cdot}(t_1) - \varphi_2^{\cdot}(t_1 - t_2)] \pm \varphi_2^{\cdot}(t_1 - t_2)$$

Second equation of each system covers the case, when the first one becomes an identity.

We shall also consider the region $D_A^{\cdot}(t_1)$ no longer as a set of terminal points of operations when the time $T \leq t_1$, but as a set of the end-points of economical trajectories with fixed time t_1 and minimum value of

$$\min\int_0^\infty |u|\,dt = \nu$$

L e m m a 6.1. When $\nu = 1$, the set of the end-points of economical trajectories represents the boundary of the region $D_A(t_1)$, while at $\nu < 1$, the boundary contracts by the factor of $1/\nu$.

To prove it, we shall first obtain, repeating exactly the reasoning of [6], the following theorem.

T h e o r e m 6.1. The necessary and sufficient condition for the economical trajectory from the point (0, 0) to the point $(x_0, x_0^{\cdot})$ with fixed t and $\nu = 1$ to exist is, that σ_1^0 and σ_2^0 are solution of the problem

$$\min_{c_1, c_2}\max_{0\leq t\leq t_1}|c_1\varphi_2(t_1 - t) + c_2\varphi_2^{\cdot}(t_1 - t)| = 1, \qquad c_1x_0 + c_2x_0^{\cdot} = -1$$

If σ_1^0 and σ_2^0 are the solution of this problem and $t^1, t^2, \ldots, t^s$ are the roots of the first equation, then the economical control is impulsive and sum of the moduli of its impulses $|\mu_1| + \ldots + |\mu_s| = 1$. Omitting the details of construction of economical trajectories as they are almost an exact repeat of those given for the operations, we shall just quote their properties.

Economical trajectory terminating at the rectilinear part of the boundary coincides, in all cases except (2) $(\mathrm{Re}\,\lambda < 0,\ t_1 > t_2)$, with the operation terminating at that point.

Economical trajectory terminating at the point on the curvilinear part of the boundary or at the point on the rectilinear part in case $\mathrm{Re}\,\lambda < 0$, $t_1 < t_2$, can be constructed as follows.

Let $(x_0, x_0^{\cdot})$ lie on the curvilinear part and let $T(x_0, x_0^{\cdot})$ be the time of the operation terminating at that point. Then, the economical trajectory follows geometrically the path of the operation, but t^1 – the instant of the first impulse $|\mu_1| = 1$, is delayed by the amount of time resulting in $t^1 = t_1 - T(x_0, x_0^{\cdot})$.

If, in the case (2) $[\mathrm{Re}\,\lambda < 0,\ t_1 < t_2]$, $(x_0, x_0^{\cdot})$ lies on the rectilinear part of the boundary $D(t_2)$ and $t_1 > t_2$, then geometrically operation coincides with the economical trajectory, but $t^1 = t_1 - t_2$.

In case $\lambda_{12} = \pm i\omega$, $t_1 > \pi/\omega$, economical trajectories are no longer single-valued, and an economical trajectory terminating at the point $(x_0, x_0^{\cdot})$ can have its first impulse either at $t^1 = t_1 - T(x_0, x_0^{\cdot})$ or at $t^1 = t_1 - T(x_0, x_0^{\cdot}) - \pi/\omega$. Such cases may occur when $\mathrm{Re}\,\lambda_1 = 0$, $\lambda_2 > 0$, or when $\mathrm{Re}\,\lambda_2 = 0$, $\lambda_1 > 0$, or $\lambda_1 = \lambda_2 = 0$. Later we shall see, that detailed analysis of these cases is not necessary.

Turning our attention to $\nu(x_0, x_0^{\cdot}, t_1)$ which represents the consumption on the economical trajectory originating at zero and which we shall consider as a function of the terminal point, we shall state some of its properties without proof.

At any fixed $t_1 > 0$, function $\nu(x_0, x_0^{\cdot}, t_1)$ exists over the whole space, is positive definite and becomes infinitely large as $x_0^2 + x_0^{\cdot 2} \to \infty$. For any given t_1, $\nu(x_0, x_0^{\cdot}, t) \geqslant \nu(x_0, x_0^{\cdot}, t_1), 0 < t \leqslant t_1$.

Geometrically it is obvious that the lines $\nu(x_0, x_0^{\cdot}, t_1) = \varkappa = \mathrm{const} > 0$ represent the boundary of $D(t_1)$ transformed $\varkappa$-times. Denoting by ψ the angle between the radius vector of the point $(x_0, x_0^{\cdot})$ and the x-axis and by θ the angle between the x-axis and the tangent to the boundary, we have

$$\frac{\partial\nu}{\partial x_0}\cos\psi + \frac{\partial\nu}{\partial x_0^{\cdot}}\sin\psi = \frac{\nu}{\sqrt{x_0^2 + x_0^{\cdot 2}}} = \mathrm{const}, \qquad \frac{\partial\nu}{\partial x_0}\cos\theta + \frac{\partial\nu}{\partial x_0^{\cdot}}\sin\theta = 0 \qquad (6.2)$$

Determinant of this system $\Delta = \sin(\psi - \theta)$ is different from zero, since the angle between the radius vector and the tangent $\psi - \theta$ is contained within the limits $0 < \psi - \theta < \pi$.

Angle θ undergoes a discontinuous change at all vertices of the boundary, consequently derivatives $\partial\nu/\partial x_0$ and $\partial\nu/\partial x_0^{\cdot}$ are also discontinuous. First formula of the system (6.2) gives a derivative in the direction of the radius vector, the second one – along the boundary.

Returning now to the initial problem, we shall prove a theorem.

T h e o r e m 6.2. If the velocity of penetration vector is directed into the region $D(t_1)$, then the time of operation $T(x_0, \ldots, x_1^{\cdot})$ is a continuous function of its variables. If, in addition, point of intersection $(x_E, x_E^{\cdot})$ of the trajectory of the point B with the boundary of $D(t_1)$ is not a vertex, then $T(x_0, \ldots, x_1^{\cdot})$ is also continuously differentiable. If, on the other hand, the velocity of penetration is tangential to the boundary or is directed outward, then the time of operation is discontinuous.

P r o o f. Time T of the operation is the smallest positive root of

$$\nu(x_B, x_B^{\cdot}, t) = 1 \qquad (6.3)$$

$$x_B = x_0 - x_1\varphi_1(t) - x_1^{\cdot}\varphi_2(t), \qquad x_B^{\cdot} = x_0^{\cdot} - x_1\varphi_1^{\cdot}(t) - x_1^{\cdot}\varphi_2^{\cdot}(t)$$

If the velocity of penetration is directed into the region, then for small Δt, we have

$$\nu(x_B(T + \Delta t), \quad x_B^{\cdot}(T + \Delta t), \quad T + \Delta t) < 1$$

This means that a continuous solution $T(x_0, x_0^{\cdot}, x_1, x_1^{\cdot})$ of Equation (6.3) exists near $x_0, x_0^{\cdot}, x_1, x_1^{\cdot}$. Differentiating (6.3), we obtain

$$\frac{\partial\nu}{\partial x_B}dx_B + \frac{\partial\nu}{\partial x_B^{\cdot}}dx_B^{\cdot} + \frac{\partial\nu}{\partial t}dt = 0 \qquad (6.4)$$

$$dx_B = (x_B^{\cdot} - x_0)dt + dx_0 - \varphi_1^{\cdot}(t)\,dx_1^{\cdot} - \varphi_2(t)\,dx_1^{\cdot} \qquad (6.5)$$

$$dx_B^{\cdot} = x_B^{\cdot\cdot}\,dt + dx_0^{\cdot} - \varphi_1(t)\,dx_1 - \varphi_2^{\cdot}(t)\,dx_1^{\cdot}$$

Partial derivatives $\partial\nu/\partial x_B$ and $\partial\nu/\partial x_B^{\cdot}$ exist and are continuous at all points except the vertices, and the coefficient of dt in (6.4) obtained with (6.5) taken into account, is less than zero. The latter follows from the requirement that the vector of the velocity of penetration is orientated in a certain direction. From this, the continuity of partial derivatives of $T(x_0, \ldots, x_1^{\cdot})$ follows.

If the velocity of penetration vector is tangent to the curvilinear part of the boundary, then such $e = \mathrm{const}$ exists, that, when $t = t_1$, then

$$x_1\varphi_1^{\cdot}(t_1) + (x_1^{\cdot} - e)\varphi_2^{\cdot}(t_1) = 0, \qquad x_1\varphi_1^{\cdot\cdot}(t_1) + (x_1^{\cdot} - e)\varphi_2^{\cdot\cdot}(t_1) = 0 \qquad (6.6)$$

which means that they are identically satisfied, and point B moves along the boundary of $D(t)$. A small displacement to the next integral line will cause a discontinuity in T.

If the velocity of penetration is colinear with the straight line segment, then a constant e exists such, that

$$-x_A^{\cdot} = \pm e\varphi_2 \pm \lambda\varphi_2^{\cdot}, \qquad -x_A^{\cdot\cdot} = \pm e\varphi_2^{\cdot} \pm \lambda\varphi_2^{\cdot\cdot} \qquad (6.7)$$

These equations confirm that the point $(x_0, x_0^{\cdot})$ lies on the envelope (5.6) since they give rise to the same set of values of λ, as (5.6.3).

Let us now assume that the velocity of penetration is directed out of the region $D(t_1)$. This can happen only when the point ($x_E, x_E^{\cdot}$) coincides with a vertex of $D(t_1)$, since otherwise t_1 would not be the first root of (6.3). If this point a_1 or b_1 is fixed with respect to the moving system, then the time is discontinuous and point $(x_0, x_0^{\cdot})$ belongs to the segment of the boundary of the type (5.4).

Moving vertex travels, with respect to a fixed system, along the integral line of $x^{\cdot\cdot} + bx^{\cdot} + cx = 0$, and the corresponding impulse is $\mu_1 = 1$ at $t^1 = 0$. If the point $(x_0, x_0^{\cdot})$ lies on this line, then it is clear that when $t = t_1 + \Delta t_1$, its velocity of penetration is tangent to the boundary, and the time of operation is discontinuous.

At $t_1 = 0$, the statement that the velocity of penetration is directed into the region $D(0)$, is meanigless. We can however say that at the points $x = x_1$, $x_1^{\cdot} - 1 \leq x^{\cdot} \leq x_1^{\cdot} + 1$ for which the limiting value of the velocity of penetration vector is directed into the region $D(t_1)$ as $t_1 \to 0$, the time of the operation is continuous, while at the points where it is either directed outward or equal to zero, the time is discontinuous. The proof of this is analogous to the previous case.

7. Having explained the properties of the function $T(x_0, x_0^{\cdot}, x_1, x_1^{\cdot})$, we shall next consider all possible distributions of roots on the complex plane. Let $x_E, x_E^{\cdot}$ be the coordinates of the point of intersection of the curve $x_B, x_B^{\cdot}$ with the boundary of $D(t_1)$, or $D(T)$. Denoting as before the time of operation from zero to the point $x, x^{\cdot}$ by $T(x, x^{\cdot})$ and the value of its first impulse by $\mu_1(x, x^{\cdot})$, the value of the first impulse in the operation from $(x_1, x_1^{\cdot})$ to $(x_0, x_0^{\cdot})$ by $\mu_1 = \mu_1(x_0, x_0^{\cdot}, x_1, x_1^{\cdot})$ and the times of the first and second impulse by $t^1(x_0, x_0^{\cdot}, x_1, x_1^{\cdot})$ and $t^2(x_0, x_0^{\cdot}, x_1, x_1^{\cdot})$ respectively, we obtain

C a s e $\lambda_{12} = \beta \pm i\omega$, $\beta > 0$. Let point B be on the rectilinear segment of the boundary $D(T)$

$$t^1 = 0, \quad t^2 = T, \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) \quad \text{for } T \leqslant t_2 \qquad (7.1)$$

$$t^1 = 0, \quad t^2 = t_2, \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) \quad \text{for } T > t_2 \qquad (7.2)$$

If B falls on the curvilinear part of the boundary, then

$$t^1 = T - T(x_E, x_E^{\cdot}), \quad t^2 = T, \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) = \pm 1 \qquad (7.3)$$

Indeed we find, that the operation is, necessarily, an economical trajectory utilizing the whole reserve (in the following, construction of operations will be based on this fact).

C a s e $\lambda_{12} = \beta \pm i\omega$, $\beta < 0$. If $T \leq t_2$, then for the point B on the rectilinear part of the boundary, we have

$$t^1 = 0, \quad t^2 = T, \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) \qquad (7.4)$$

On the curved part when $T \leq t_2$, we have

$$t^1 = T - T(x_B, x_B^{\cdot}), \quad t^2 = T, \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) \qquad (7.5)$$

On the straight part when $T > t_2$, we have

$$t^1 = T - t_2, \quad t^2 = T, \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) \tag{7.6}$$

Case $\lambda_1 > 0$, $\lambda_2 < 0$. The boundary consists of straight line segments

$$t^1 = 0, \ t^2 = T, \mu_1 = \mu_1(x_E, x_E^{\cdot}) \tag{7.7}$$

Case $\lambda_{12} = \pm i\omega$. On the straight part we have

$$t^1 = 0, \ t^2 = T, \ \mu_1 = \mu_1(x_E, x_E^{\cdot})$$

On the curved part

$$t^1 = T - T(x_E, x_E^{\cdot}), \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) = \pm 1 \quad \text{for } T < \pi/\omega \tag{7.8}$$

When $2\pi/\omega > T > \pi/\omega$, two operations are possible

$$t^1 = T - T(x_E, x_E^{\cdot}), \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) \tag{7.9}$$

$$t^1 = T - T(x_E, x_E^{\cdot}) - \pi/\omega, \mu_1 = \mu_1[x_E(T - \pi/\omega), x_E^{\cdot}(T - \pi/\omega)]$$

Case $\lambda_1 > 0$, $\lambda_2 = 0$. In this case point B can only be found on the straight line portion $(a_2 b_1)$ and $(b_2 a_1)$. Velocity of penetration can also be directed inward only on the open segment $(a_2 b_1)$ of $(b_2 a_1)$

Therefore, as already remarked in Section 6, the properties of economical trajectories which, when their terminal point falls on the segment $a_1 a_2$ may become no longer unique, are not particularly interesting

$$t^1 = 0, \quad t^2 = T, \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) \tag{7.10}$$

Case $\lambda_2 < 0$, $\lambda_1 = 0$. We have the repeat of case (7.5)

$$t^1 = 0, \quad t^2 = T, \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) \tag{7.11}$$

Case $\lambda_1 = \lambda_2 = 0$.

$$t^1 = 0, \quad t^2 = T, \quad \mu_1 = \mu_1(x_E, x_E^{\cdot}) \tag{7.12}$$

Formulas (7.1) to (7.12) describe the functions μ_1, t^1 and t^2 in terms of the functions $T(x, x^{\cdot})$ and $\mu_1(x, x^{\cdot})$ investigated in detail at the beginning of this paper.

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